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Notes on certain inhomogeneous three term recurrences

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Abstract

The general three term recurrence with constant coefficients and a constant or alternating constant inhomogeneous term is considered. The generating function is given, and the equivalent four term homogeneous recurrence following from it is stated.

The following two types of inhomogeneous three term recurrences will be considered.

$$\textbf{A)}\quad a_n \;=\; c\,a_{n-1} + d\,a_{n-2} + k, \quad n = 2,3,..., \; a_0 = a,\; a_1 = b\,, \tag{1}$$

$$\textbf{B)}\quad b_n \;=\; c\,b_{n-1} + d\,b_{n-2} + (-1)^n\,k, \; n = 2,3,..., \; b_0 = a,\; b_1 = b\,. \tag{2}$$

These sequences will be called $\{A(a,b;c,d;k;n)\}_0^\infty$, and $\{B(a,b;c,d;k;n)\}_0^\infty$, respectively. Such sequences have been considered by Gary Detlefs to whom the author is grateful for email exchange and a reading of this paper.

The ordinary generating function (*o.g.f.*) of these sequences are denoted by $A(x) \equiv A(a,b;c,d;k;x) := \sum_{n=0}^{\infty} a_n\,x^n$, and $B(x) \equiv B(a,b;c,d;k;x) := \sum_{n=0}^{\infty} b_n\,x^n$, respectively. They are considered in the framework of formal power series, and will be given in *Proposition 1*, resp.*Proposition 5*. In order to formulate them some definitions and lemmata are given first.

Definition 1. Generalized Fibonacci sequence $\{\mathbf{U(c,d;n)}\}_{\mathbf{0}}^{\infty}$.

The general $(c\,d)-$*Fibonacci* sequence $U_n \equiv U(c,d;n)$ with standard inputs is

$$U_n = c\,U_{n-1} + d\,U_{n-2}, \quad n = 2,3.\,..., \; U_{-1} = 0, U_0 = 1, \; (U_1 = c)\,. \tag{3}$$

Lemma 1. O.g.f. for $\{\mathbf{U_n}\}$

The *o.g.f.* $U(x) \equiv U(c,d;x) := \sum_{n=0}^{\infty} U_n\,x^n$ is

$$U(x) = \frac{1}{1 - c\,x - d\,x^2}\,. \tag{4}$$

Proof: $U(x) = 1 + c\,x + \sum_{n=2}^{\infty} (c\,U_{n-1} + d\,U_{n-2}\,)\,x^n = 1 + c\,x + x\,c(U(x) - 1) + x^2\,d\,U(x)$, which is solved for $U(x)$. □

Lemma 2. Explicit Binet/de Moivre formula for $\mathbf{U(c,d;n)}$.

$$U_n \equiv U(c,d;n) = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}, \text{ with } \lambda_\pm := \frac{1}{2}\,(c \pm \sqrt{c^2 + 4\,d}\,)\,. \tag{5}$$

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Proof: This is well known. Just use the characteristic equation for this recurrence and fit the inputs, using the linearity of the recurrence. It can also be derived (in the sense of formal power series) from the *o.g.f.* $U(x)$ using factorization of the denominator and partial fraction decomposition. □

Definition 2. Generalized Fibonacci sequence $\{\mathbf{F(A,B;c,d;n)}\}_{\mathbf{0}}^{\infty}$.

$f_n \equiv F(A,B;c,d;n)$ is defined by

$$f_n \,=\, c\, f_{n-1} \,+\, d\, f_{n-2}\,,\;\; f_0 \,=\, A,\;\; f_1 \,=\, B\;. \tag{6}$$

Lemma 3.

$$f_n \,=\, A\, U_n \,+\, (B \,-\, c\, A)\, U_{n-1}\;,\;\; n \in \mathbb{N}_0\;. \tag{7}$$

Proof: From the linearity of the $f-$recurrence and *Definition 1*, after fitting the two inputs. □

Lemma 3. O.g.f. for $\{\mathbf{F(A,B;c,d;n)}\}_{\mathbf{0}}^{\infty}$

The *o.g.f.* $F(x) \,\equiv\, F(A,B;c,d;x) \,:=\, \sum_{n=0}^{\infty} f_n\, x^n$ is

$$F(x) \,=\, \frac{A \,+\, (B \,-\, c\, A)}{1 \,-\, c\, x \,-\, d\, x^2}\;. \tag{8}$$

Proof: Similar to the proof of *Lemma 1.* □

Proposition 1. O.g.f. for $\{\mathbf{A(a,b;c,d;k;n)}\}_{\mathbf{0}}^{\infty}$

The *o.g.f* $A(x) \,\equiv\, A(a,b;c,d;k;x) \,:=\, \sum_{n=0}^{\infty} a_n\, x^n$ is

$$\begin{aligned} A(x) &= \left(a \,+\, (b \,-\, a\, c)\, x \,+\, x^2\, k\, \frac{1}{1 \,-\, x}\right) U(c,d;x) && (9)\\ &= \frac{1}{1 \,-\, x}\, U(c,d;x)\, \left(a \,+\, (b \,-\, a\,(1 \,+\, c))\, x \,-\, (b \,-\, k \,-\, a\, c)\, x^2\right) && (10)\\ &= \left(a \,+\, (b \,-\, a\,(1 \,+\, c))\, x \,-\, (b \,-\, k \,-\, a\, c)\, x^2\right) \frac{1}{1 \,-\, (1 \,+\, c)\, x \,-\, (d \,-\, c)\, x^2 \,+\, d\, x^3}\;. && (11) \end{aligned}$$

Proof:

$$\begin{aligned} A(x) &= a \,+\, b\, x \,+\, \sum_{n=2}^{\infty} c\; a_{n-1}\, x^n \,+\, \sum_{n=2}^{\infty} d\; a_{n-2}\, x^n \,+\, k\, \sum_{n=2}^{\infty} x^n \\ &= a \,+\, b\, x \,+\, c\, x\,(A(x) \,-\, a) \,+\, d\, x^2\, A(x) \,+\, k \left(\frac{1}{1 \,-\, x} \,-\, (1+x)\right). \end{aligned}$$

Solving for $A(x)$ produces the first formula. The other two follow immediately. □

Proposition 2.

$$a_n \,\equiv\, A(a,b;c,d;k;n) \,=\, F(a,b;c,d;n) \,+\, k \sum_{l=0}^{n-2} F(1,c;c,d;l)\;. \tag{12}$$

Proof:

From the first formula of *Proposition 1* and the *o.g.f.* $F(x)$ in *Lemma 3*. The sum has to be put to zero for $n = 0$ and $n = 1$. □

Definition 3. $\{\mathbf{\hat{F}(A,B,C;c,d;n)}\}_{\mathbf{0}}^{\infty}$ sequence

$\hat{f}_n \,\equiv\, \hat{F}(A,B,C;c,d;n)$ is defined by

$$\hat{f}(n) \,=\, c\, \hat{f}_{n-1} \,+\, d\, \hat{f}_{n-2}\,,\;\; n \,=\, 3,4,...,\;\; \hat{f}_0 \,=\, A,\;\; \hat{f}_1 \,=\, B,\;\; \hat{f}_2 \,=\, C\;. \tag{13}$$

Lemma 4. O.g.f. for $\hat{\mathbf{F}}(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{c},\mathbf{d};\mathbf{x})$

The *o.g.f.* $\hat{F}(x) \equiv \hat{F}(A,B,C;c,d;x) := \sum_{n=0}^{\infty} \hat{f}_n\, x^n$ is

$$\hat{F}(x) = \left(A + (B - c\,A)\,x + (C - c\,B - d\,A)\,x^2\right)\,U(c,d;x)\ . \tag{14}$$

Proof.

$$\begin{aligned}\hat{F}(x) &= A + B\,x + C\,x^2 + \sum_{n=3}^{\infty} (c\,\hat{f}_{n-1} + d\,\hat{f}_{n-2})\,x^n \qquad (15)\\ &= A + B\,x + C\,x^2 + c\,x\,(\hat{F}(x) - (A + B\,x)) + d\,x^2\,(\hat{F}(x) - A)\,, \qquad (16)\end{aligned}$$

solving for $\hat{F}(x)$. □

Corollary 1.

$\hat{f}_n = A\,U(c,d;n) + (B - c\,A)\,U(c,d;n-1) + (C - c\,B - d\,A)\,U(c,d;n-2)\ ,\ n \in \mathbb{N},\ \hat{f}_0 = A.$

Proof. From the *o.g.f.* $\hat{F}(x)$ after comparing coefficients of x^n. □

Proposition 3.

$$a_n \equiv A(a,b;c,d;k;n) = \sum_{l=0}^{n} \hat{F}(a, b-a; k-b+c\,b+a\,d; c,d;n)\ . \tag{17}$$

Proof. With the second formula for $A(x)$ in *Proposition 1.* The partial sums appear because of the factor $\frac{1}{1-x}$ in the *o.g.f.* □

Proposition 4.

The inhomogeneous recurrence for $a_n \equiv A(a,b;c,d;k;n)$ is equivalent to the following four term homogeneous recurrence relation.

$$\begin{aligned}a_n &= (1 + c)\,a_{n-1} + (d - c)\,a_{n-2} - d\,a_{n-3}\,,\ n = 3,4,...,\\ &\quad \text{with the inputs } a_0 = a,\ a_1 = b,\ a_2 = c\,b + d\,a + k\ . \qquad (18)\end{aligned}$$

This has been observed by Gary Detlefs.

Proof. The *o.g.f.* $G(x) := \sum_{n=0}^{\infty} a_n\,x^n$ of this sequence is seen to coincide with the third formula for $A(x)$ given in *Proposition 1*, due to

$$\begin{aligned}G(x) &= a + b\,x + (c\,b + d\,a + k)\,x^2 + \sum_{n=3}^{\infty} ((1 + c)\,a_{n-1} + (d - c)\,a_{n-2} - d\,a_{n-3})\,x^n\\ &= a + b\,x + (c\,b + d\,a + k)\,x^2 + (1 + c)\,x\,(G(x) - (a + b\,x)) + (d - d)\,x^2\,(G(x) - a) - d\,x^3\,G(x).\end{aligned}$$

Solving for $G(x)$ shows that it is identical with $A(x)$ from *Proposition 1.* □

Analogous results are derived for the family of sequences $\{B_n \equiv B(a,b;c,d;k;n)\}$ with alternating inhomogeneous $k-$term. We only quote the results without proofs. They follows the same steps as above.

Proposition 5. O.g.f. for $\{\mathbf{B}(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d};\mathbf{k};\mathbf{n})\}_{\mathbf{0}}^{\infty}$

The *o.g.f* $B(x) \equiv B(a,b;c,d;k;x) := \sum_{n=0}^{\infty} b_n\,x^n$ is

$$\begin{aligned}B(x) &= \left(a + (b - a\,c)\,x + x^2\,k\,\frac{1}{1 + x}\right)U(c,d;x) \qquad (19)\\ &= \frac{1}{1 + x}\,U(c,d;x)\,\left(a + (b + a\,(1 - c))\,x + (b + k - a\,c)\,x^2\right) \qquad (20)\\ &= \left(a + (b + a\,(1 - c))\,x + (b + k - a\,c)\,x^2\right)\,\frac{1}{1 - (c - 1)\,x - (d + c)\,x^2 - d\,x^3}\ . \qquad (21)\end{aligned}$$

Proposition 6.

$$b_n \equiv B(a,b;c,d;k;n) = F(a,b;c,d;n) + k \sum_{l=0}^{n-2} (-1)^{n-l}\, F(1,c;c,d;l) \ . \tag{22}$$

Proposition 7.

$$b_n \equiv B(a,b;c,d;k;n) = \sum_{l=0}^{n} (-1)^{n-l}\, \hat{F}(a, b+a; k+b+c\,b+a\,d; c,d;l) \ . \tag{23}$$

Proposition 8.

The inhomogeneous recurrence for $b_n \equiv B(a,b;c,d;k;n)$ is equivalent to the following four term homogeneous recurrence relation.

$$\begin{aligned} b_n &= (c-1)\, b_{n-1} + (d+c)\, b_{n-2} + d\, b_{n-3}\,, \ n = 3,4,..., \\ &\text{with the inputs } b_0 = a, \ b_1 = b, \ b_2 = c\,b + d\,a + k \ . \end{aligned} \tag{24}$$

This has been observed by Gary Detlefs.

Note added, Dec 21 2010.

Provided $d = 1 + c$ every A-sequence coincides with some B-sequence. Similarly, if $d = 1 - c$ every B-sequence coincides with some A-sequence. These relationships are, for $n \geq 0$:

$$A(a,b;c,1+c;k;n) = B(a,b;c+2,-(1+c);k-2\,b+2\,a(1+c);n) \ , \tag{25}$$

$$B(a,b;c,1-c,k;n) = A(a,b;c-2,c-1;k+2\,b+2\,a(1-c);n) \ . \tag{26}$$

The proof is simple when one compares the denominators and numerators of the two *o.g.f.*s. Thanks go to G. Detlefs for bringing this point up.

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Concerned, among others, with OEIS sequences (this list of $\{A(a,b;c,d;k;n)\}$ type sequences has been adapted from a list compiled by Gary Detlefs. In his list the data [a,b:c,d:k] is given with additional remarks.)

A000012, A000035, A000124, A000217, A000290, A000326, A000340, A000384, A000566, A000567, A000975, A000975, A001106, A001107, A001108, A001477, A001550, A001576, A001595, A002061, A003462, A004146, A005322, A005449, A005578, A005665, A007598, A014105, A014335, A024537, A024551, A027941, A027961, A033113, A033144, A033538, A033539, A033539, A035344, A045944, A047926, A048739, A049651, A049652, A051049, A051624, A051682, A051865, A051866, A051867, A051868, A051869, A051870, A051871, A051872, A051873, A051874, A051875, A051876, A052948, A052986, A053142, A054493, A055269, A060188, A061278, A069403, A073724, A074502, A074506, A074507, A074508, A074509, A074511, A074513, A074516, A074520, A074522, A074523, A077846, A082574, A082585, A084170, A084640, A089817, A092184, A098790, A099232, A100227, A100227, A102871, A103820, A109164, A111314, A111721, A147875, A155602, A155603, A155617, A157728, A159290, A160156, A174192, A174192.