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1) O.g.f. for elementary symmetric functions:

$$\Sigma(t) \;:=\; \prod_{j=1}^{n\,(\infty)} (1 - x_j\, t) \;=\; \sum_{k=0}^{\infty} (-1)^k\, \sigma_k\, t^k \;.$$

2) O.g.f. power sums $p_r \;:=\; \sum_{j=1}^{n} (x_j^r)$:

$$P(t) \;=\; \sum_{r=1}^{\infty} p_r\, t^r \;=\; -t\,(ln\,\Sigma(t))' \;.$$

3) Multinomial formula involving $M2$ (A-St, p.823 called $(n, a_1, ..., a_n)^*$ there, but M2 on p. 831-2.)

$$\frac{1}{m!} \left(\sum_{k=1}^{\infty} \frac{x_k}{k}\, t^k \right)^m \;=\; \sum_{n=m}^{\infty} \frac{t^n}{n!} \sum M2(n, \vec{a})\, x_1^{a_1} \cdots x_n^{a_n}$$

where the second sum is over $\sum a_j \;=\; m$, $\sum j\, a_j \;=\; n$, with $a_j \in \mathbb{N}_0$ and the number of parts $m \in \{1, ..., n\}$ of the partitions of n.

Proposition:

$$(-1)^n\, n!\, \sigma_n \;=\; \sum_{m=0}^{n} (-1)^m \sum_{\vec{a} \in Pa(n,m)} M2(n, \vec{a}) \prod_{j=1}^{n} p_j^{a_j} \;.$$

$Pa(n,m)$ stands for the set of partitions of n with m parts.

Proof:

From **2)**: $\frac{d}{dt} ln \Sigma(-t) \;=\; -\frac{1}{t}\, P(-t) \;=\; p_1 \,-\, p_2\, t \,+\, p_3\, t^2 - + ...$

Integrated:

$$ln\,\Sigma(-t) \;=\; -(p_1\,(-t) \,+\, p_2\, \frac{(-t)^2}{2} \,+\, p_3\, \frac{(-t)^3}{3} \,+\, ...$$

$$\Sigma(-t) \;=\; exp(-(p_1\,(-t) + p_2\, \frac{(-t)^2}{2} + p_3\, \frac{(-t)^3}{3} + ...)) \;=\; \sum_{m}^{\infty} \frac{(-1)^m}{m!}\, (p_1\,(-t) + p_2\, \frac{(-t)^2}{2} + ...)^m.$$

Use of **3)** yields

$$\sum_{m=0}^{\infty} (-1)^m \sum_{n=m}^{\infty} \frac{(-t)^n}{n!} \sum M2(n, \vec{a}) \prod_{j=1}^{n} p_j^{a_j} \;.$$

Rearrange the double sum (not worrying about convergence) yields:

$$\Sigma(-t) \;=\; \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \sum_{m=1}^{n} (-1)^m \sum_{\vec{a} \in Pa(n,m)} M2(n, \vec{a}) \prod_{j=1}^{n} p_j^{a_j} \;.$$

Pick from this the coefficient of $\frac{t^n}{n!}$ to find the proposition. □

Literature:

V. Krishnamurthy, *Combinatorics*, Ellis Horwood, New York, 1986., p.60, Exercise 5*, first eq.